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Perturbation Theory for Volterra Integrodifferential Systems

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I. INTRODUCTION

The purpose of this paper is to study the asymptotic behavior of solutions of a Volterra integrodifferential system of the form

$$x'(t) = f(t) + A(t)\,x(t) + \int_0^t B(t, s)\,x(s)\,ds + (gx)(t), \qquad x(0) = x_0 \tag{N}$$

where $0 \leqslant t < \infty$, A and B are $n \times n$ matrices and $g\varphi$ is a "small" nonlinear functional. The system will be studied as a perturbation of the linear system

$$y'(t) = f(t) + A(t)\,y(t) + \int_0^t B(t, s)\,y(s)\,ds, \qquad y(0) = x_0\,. \tag{L}$$

By a solution of (N) or (L) on an interval $0 \leqslant t \leqslant \sigma$ we shall mean an absolutely continuous function $x(t)$ or $y(t)$ which solves the equation a.e. As examples of "small" pertubations we have in mind "higher order" terms of the form

$$(g\varphi)(t) = \int_0^t c(t, s)\,\varphi^2(s)\,ds \qquad \text{or} \qquad \varphi(t) \int_0^t c(t, s)\,\varphi(s)\,ds$$

which occur in reactor dynamics.

Assuming that system (L) is stable in some sense, we shall prove that, for x_0 and f small, equation (N) has a corresponding stability property. Results

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of a similar nature have recently been obtained for Volterra integral equations by Miller [1], Miller, Nohel, and Wong [2], and Grossman [3]. The results in Section III below are generalizations of earlier results of the first author [3].

The main tools in our analysis are the "variation of constants" formula obtained in Section II and the idea of admissibility of integral operators (see Lemma 3 below). The variation of constants formula used here is similar to the usual formulas for ordinary and functional differential equations; see, e.g., Banks [4]. The idea of admissibility originated with Massera and Schaeffer [5]. It has recently been applied to integral equations by Corduneanu [6, 7] and Antosiewicz [8].

II. A Variation of Constants Formula

Let R^+ be the real half line $0 \leqslant t < \infty$ and let E^n denote the real or complex n-dimensional space of column vectors with the usual norm $|z|$. Let I denote the identity in any space under consideration. Given any set $S \subset E^n$ and any p in the range $1 \leqslant p < \infty$, the set $LL^p(S)$ will be the set of all measurable functions f on S such that the seminorms

$$\|f\|_T = \left\{\int_T |f(x)|^p \, dx\right\}^{1/p} \tag{1}$$

are finite for all compact subsets $T \subset S$. The set of all seminorms of the form (1) generate a metric topology on $LL^p(S)$.

If $A(t)$ and $B(t, s)$ are the functions given in equations (L) then formally define

$$\Psi(t, s) = A(t) + \int_s^t B(t, u) \, du \qquad (0 \leqslant s \leqslant t < \infty) \tag{2}$$

and

$$R(t, s) = I + \int_s^t R(t, u) \, \Psi(u, s) \, du \qquad (0 \leqslant s \leqslant t < \infty) \tag{3}$$

and let $B(t, s) = \Psi(t, s) = R(t, s) = 0$ if $s > t \geqslant 0$. Then $R(t, s)$ is a formal solution of the adjoint equation

$$\frac{\partial}{\partial s} R(t, s) = -R(t, s) \, A(s) - \int_s^t R(t, u) \, B(u, s) \, du, \qquad R(t, t) = I \tag{A}$$

on the interval $0 \leqslant s \leqslant t$.

LEMMA 1. *Assume*

$$\text{(H1)} \qquad A \in LL^1(R^+), \qquad B \in LL^1(R^+ \times R^+).$$

Then the function $R(t, s)$ *as defined in* (3) *exists on* $0 \leqslant s \leqslant t$ *and is continuous in* (t, s). $(\partial/\partial s)\, R(t, s)$ *exists a.e. on* $0 \leqslant s \leqslant t$, *is in* $LL^1(R^+ \times R^+)$, *and satisfies equation* (A) *on* $0 \leqslant s \leqslant t$, *for each* $t > 0$. *Moreover, given any vector* x_0 *and any function* f *in* $LL^1(R^+)$ *equation* (L) *is equivalent to the system*

$$y(t) = R(t, 0)\, x_0 + \int_0^t R(t, s) f(s)\, ds. \tag{4}$$

Proof. Since $\Psi(t, s)$ is continuous in s for each fixed t, the existence of $R(t, s)$ on $0 \leqslant s \leqslant t$ is trivial. It also follows easily that for each fixed t, $(\partial/\partial s)\, R(t, s)$ exists and satisfies (A).

Since B is LL^1 on $0 \leqslant s \leqslant t < \infty$, we have

$$|\psi(t, s)| \leqslant |A(t)| + \int_0^t |B(t, u)|\, du = \alpha(t)$$

and $\alpha \in LL^1(R^+)$. In (3) let $y(\tau) = R(t, t - \tau)$, apply the Gronwall inequality, then set $\tau = t - s$ in $|y(\tau)| = |R(t, t - \tau)|$. This yields the estimate

$$|R(t, s)| \leqslant \alpha_0(t) = 1 + \left(\int_0^t \alpha(u)\, du\right) \exp\left(\int_0^t \alpha(u)\, du\right), \tag{5}$$

which implies that R is $LL^1(R^+ \times R^+)$. Using this fact in (A) it is apparent that $(\partial/\partial s)\, R(t, s)$ is $LL^1(R^+ \times R^+)$ and that

$$\begin{aligned} \left| \frac{\partial}{\partial s} R(t, s) \right| &\leqslant \alpha_0(t)\, |A(s)| + \int_s^t \alpha_0(t)\, |B(u, s)|\, du \\ &\leqslant \alpha_0(T) \left(A(s) + \int_s^T B(u, s)\, du\right) \end{aligned} \tag{6}$$

if $0 \leqslant s \leqslant t \leqslant T$. Continuity of $R(t, s)$ in t (for s fixed) then follows using (5) and dominated convergence. (6) implies that $R(t, s)$ is continuous in s uniformly for $0 \leqslant s \leqslant t \leqslant T$. Continuity in t and uniform continuity in s imply continuity in the pair (t, s).

Now, let x_0 and f be fixed as in the lemma. Let $y(t)$ be a solution of (L) on an interval $0 \leqslant t \leqslant T$. For any fixed $t > 0$, integration by parts yields

$$\int_0^t \left\{R(t, s)\, y'(s) - \left(\frac{\partial}{\partial s} R(t, s)\right) y(s)\right\} ds = R(t, s)\, y(s) \Big|_{s=0}^{s=t}.$$

Since $R(t, t) = I$ and y satisfies (L), this may be rearranged in the form

$$y(t) - R(t, 0)\, y_0 - \int_0^t R(t, s) f(s) = \int_0^t \left\{ \frac{\partial}{\partial s} R(t, s) + R(t, s)\, A(s) \right\} y(s)\, ds + \int_0^t \int_0^s R(t, s)\, B(s, u)\, y(u)\, du\, ds. \tag{7}$$

Note that $| R(t, s)\, B(u, s)| \leqslant \alpha_0(T) \, | B(u, s)| \in L^1$ over $0 \leqslant s \leqslant u \leqslant t \leqslant T$. Using Fubini's theorem, the last term of (7) may be put in the form $\int_0^t \int_S^t \{R(t, u)\, B(u, s)\, du\}\, y(s)\, ds$. Thus the right side of (7) has the form

$$\int_0^t \left\{ \frac{\partial}{\partial s} R(t, s) + R(t, s)\, A(s) + \int_s^t R(t, u)\, B(u, s)\, du \right\} y(s)\, ds.$$

By (A), this term is zero. Moreover, if $y(t)$ solves (4) on an interval $0 \leqslant t \leqslant \sigma$, then the calculation is easily reversed to see that $y(t)$ solves (L). Q.E.D.

LEMMA 2. *If $A(t) \equiv A$ is constant and $B(t, s) = B(t - s)$ is of convolution type with $B(t)$ in $LL^1(R^+)$, then* (H1) *is true, the functions $\Psi(t, s) = \Psi(t - s)$ and $R(t, s) = R(t - s)$ are of convolution type, and equation* (A) *takes the form*

$$R(t) = I + \int_0^t R(t - s) \left\{ A + \int_0^s B(\omega)\, d\omega \right\} ds. \tag{8}$$

Proof. In this case

$$\psi(t, s) = A + \int_s^t B(t - \omega)\, d\omega = A + \int_0^{t-s} B(\omega)\, d\omega.$$

Thus if $R(t)$ solves (8), then

$$R(t - s) = I + \int_s^t R(t - u) \left\{ A + \int_0^{u-s} B(\omega)\, d\omega \right\} du.$$

Differentiating with respect to s we see that $R(t - s)$ solves (A) on $0 \leqslant s \leqslant t$. By uniqueness $R(t - s) = R(t, s)$. Q.E.D.

We now proceed to the nonlinear equation. For this we impose the following condition:

(H2) For each pair φ_1 and φ_2 in $LL^1(R^+)$ and each $T > 0$, if $\varphi_1(t) = \varphi_2(t)$ a.e. on $0 \leqslant t \leqslant T$, then $g\varphi_1(t) = g\varphi_2(t)$ a.e. on $0 \leqslant t \leqslant T$.

THEOREM 1. *Suppose* (H1) *and* (H2) *are true and that g maps*

$LL^1(R^+) \to LL^1(R^+)$. Then on each finite interval $0 \leqslant t \leqslant \sigma$, a function $x(t)$ solves (N) *if and only if $x(t)$ solves the equation*

$$x(t) = R(t, 0)\, x_0 + \int_0^t R(t, s) f(s)\, ds + \int_0^t R(t, s)\, (gx)\, (s)\, ds \tag{9}$$

on $0 \leqslant t \leqslant \sigma$.

Proof. If $x(t)$ solves (N) on $0 \leqslant t \leqslant \sigma$, then define $f_0(t) = f(t) + (gx)\,(t)$ and apply (4). This yields (9). The process may be reversed to see that (9) implies (N). Q.E.D.

Assumption (H2) is variously described by saying that g is nonanticipative, causal, or physically realizable. If this assumption is dropped equations (N) and (9) are still equivalent on the interval $0 \leqslant t < \infty$.

THEOREM 2. *Suppose* (H1) *is true and g maps $LL^1(R^1) \to LL^1(R^+)$. Then a function $x(t)$ solves* (N) *on $0 \leqslant t < \infty$ if and only if it solves* (9) *there.*

The proof is similar to the proof of Theorem 1. In a similar way one can also prove the following:

THEOREM 3. *Suppose* (H1) *is true and $\mathscr{F}$ is a linear subspace of $LL^1(R^+)$. For any f in $LL^1(R^+)$ define*

$$(\rho f)\,(t) = \int_0^t R(t, s) f(s)\, ds \qquad (0 \leqslant t < \infty). \tag{10}$$

If ρ maps $\mathscr{F} \to \mathscr{F}$ and g maps $\mathscr{F} \to \mathscr{F}$, then equations (N) *and* (9) *are equivalent in the sense that a function $x(t)$ in $\mathscr{F}$ solves* (N) *on $0 \leqslant t < \infty$ if and only if it solves* (9) *on the same interval.*

We recall that a Frechet space is a complete metric space such that addition and scalar multiplication are continuous and the metric d is additively invariant; i.e., $d(x, y) = d(x - y, 0)$ for all x and y in the space. The spaces $LL^p(s)$ are examples of Frechet spaces.

DEFINITION 1. Let $\mathscr{F}$ be a Frechet subspace of $LL^1(R^+)$ with metric d. Then the metric topology on $\mathscr{F}$ is said to be stronger than the topology on $\mathscr{F}$ inherited from $LL^1(R^+)$ if and only if x_n, $x \in \mathscr{F}$, and $d(x_n, x) \to 0$ as $n \to \infty$ imply that $x_n \to x$ in $LL^1(R^+)$.

LEMMA 3. *Let $\mathscr{F}$ be a Frechet subspace of $LL^1(R^+)$ with a stronger topology. If* (H1) *is true and if $\rho(\mathscr{F}) \subset \mathscr{F}$, then ρ is continuous as a mapping from $\mathscr{F} \to \mathscr{F}$.*

Proof. We first show that $\rho : \mathscr{F} \to \mathscr{F}$ is closed. Let x_n, x and y be in $\mathscr{F}$

with $x_n \to x$ and $\rho x_n \to y$. Since $\mathscr{F}$ has a stronger topology, $x_n \to x$ and $\rho x_n \to y$ in $LL^1(R^+)$. Since $R(t, s)$ is continuous in (t, s) for $0 \leqslant s \leqslant t < \infty$,

$$\int_0^\sigma |\rho x_n(t) - \rho x(t)|\, dt \leqslant \int_0^\sigma \int_0^t |R(t, s)|\, |x_n(s) - x(s)|\, ds\, dt$$
$$= \int_0^\sigma \left(\int_s^\sigma |R(t, s)|\, dt \right) |x_n(s) - x(s)|\, ds \to 0$$

as $n \to \infty$ for each $\sigma > 0$. Thus $\rho x_n \to \rho x$ in $LL^1(R^+)$. By uniqueness of limits $\rho x = y$. Since ρ is defined on all of $\mathscr{F}$, the lemma follows from the closed graph theorem. Q.E.D.

As examples of the spaces $\mathscr{F}$ in Lemma 3, we have in mind any of the following:

I. $C(R^+)$ with the topology of uniform convergence on compact subsets of R^+.

II. $LL^p(R^+)$ with L^p convergence on compact subsets of R^+.

III. $BC(R^+) = \{\varphi : \varphi$ is bounded and continuous on $R^+\}$ with the sup norm.

IV. $BC_0(R^+) = \{\varphi \text{ in } BC(R^+) : \varphi(t) \to 0 \text{ as } t \to \infty\}$.

V. $L^p(R^+)$ with the usual L^p norm.

VI. $L^p(R^+) \cap BC_0(R^+)$ with the norm $\| \varphi \| = \| \varphi \|_{L^p} + \| \varphi \|_{BC}$.

III. Perturbation Theorems

We shall now study the behavior of solutions of equation (N) or equivalently of Eq. (9) when the functional g is small in the following sense:

Definition 2. Let B be a Banach subspace of $LL^1(R^+)$ with a stronger topology and let g map $B \to B$. Then g is of higher order with respect to B if and only if $g(0) = 0$, and for each $\epsilon > 0$, there exists a $\delta > 0$ such that $\| g\varphi_1 - g\varphi_2 \| \leqslant \epsilon \| \varphi_1 - \varphi_2 \|$ when φ_1 and φ_2 are in B and $\| \varphi_1 \|, \| \varphi_2 \| \leqslant \delta$.

Theorem 4. *Suppose g is of higher order with respect to B and* (H1) *is true. If f is in B, $R(t, 0)$ is in B and $\rho(B) \subset B$, then for each $\epsilon > 0$, there exists a number $\eta > 0$ such that if $| x_0 | \leqslant \eta$ and $\| f \| \leqslant \eta$, then equation* (N) *has a unique solution x in B with $\| x \| \leqslant \epsilon$.*

Proof. For any φ in B define

$$(T\varphi)(t) = R(t, 0)\, x_0 + \int_0^t R(t, s)\, \{f(s) + (g\varphi)(s)\}\, ds \qquad (0 \leqslant t < \infty).$$

By Lemma 3 the map ρ defined by (10) is continuous on B. That is

$$\| \rho \|_B = \sup\{\| \rho\varphi \|_B : \| \varphi \|_B = 1\} < \infty.$$

Since g is of higher order in B, there exists a $\delta > 0$ such that

$$\| g\varphi_1 - g\varphi_2 \| \leqslant \frac{\| \varphi_1 - \varphi_2 \|}{2 \| \rho \|} \qquad \text{if} \qquad \| \varphi_1 \| \quad \text{and} \quad \| \varphi_2 \| \leqslant \delta.$$

Given $\epsilon > 0$, define $\epsilon_0 = \min\{\delta, \epsilon\}$,

$$\eta = \min \left\{\delta, \frac{\epsilon_0}{4 \| R(t, 0)\|}, \frac{\epsilon_0}{4 \| \rho \|}\right\} \qquad \text{and} \qquad S(0, \epsilon_0) = \{\varphi \in B : \| \varphi \| \leqslant \epsilon_0\})$$

For any φ in $S(0, \epsilon_0)$, $T\varphi$ is in B and

$$\| T\varphi \| \leqslant \| R(t, 0)\| \, | x_0 | + \| \rho \| \, \| f \| + \| \rho \| \, \| g\varphi \| \leqslant \frac{\epsilon_0}{4} + \frac{\epsilon_0}{4} + \frac{\| \varphi \|}{2} \leqslant \epsilon_0 .$$

Similarly, if $\varphi_1, \varphi_2 \in S(0, \epsilon_0)$, then

$$\| T\varphi_1 - T\varphi_2 \| \leqslant \| \rho \| \, \| g\varphi_1 - g\varphi_2 \| \leqslant \frac{\| \varphi_1 - \varphi_2 \|}{2}.$$

Thus $T : S(0, \epsilon_0) \to S(0, \epsilon_0)$ is a contraction. Q.E.D.

If we assume

(H3) $A(t) \equiv A$ is constant and $B(t, s) = B(t - s)$ with $B(t)$ in $LL^1(R^+)$, then (H1) is true by Lemma 2. Moreover $R(t, s) = R(t - s)$ and $R(t)$ is in $C(R^+)$. Therefore the operator ρ always maps $C(R^+)$ or $LL^p(R^+)$ into itself. If B is any of the spaces III-VI listed at the end of Section II, then $R \in L^1(R^+)$ is sufficient to ensure that $\rho : B \to B$.

Properties of $R(t)$ can often be verified using transform theory. For example, if $B(t) \equiv 0$, then $R(t) = \exp(At)$ and R is $L^1(R^+)$ if the roots of A are in the left plane. Levin and Nohel [9] use a Tauberian theorem for Laplace transforms in order to study a linear equation of the form (L). Their conditions ensure that $R(t)$ and $R'(t)$ are both of class $BC_0(R^+) \cap L^p(R^+)$ for all $1 \leqslant p < \infty$.

If $R(t) \in L^1(R^+)$ and $\psi(t) \in L^\infty(R^+)$, then equation (8) may be used to show that R is uniformly continuous on R^+. Thus $R(t) \to 0$ as $t \to \infty$ and $R \in BC_0(R^+)$. If R is in $L^p(R^+)$ and $R' \in L^q(R^+)$ where $1 < p < \infty$ and $1/p + 1/q = 1$, then R is again uniformly continuous and belongs to $BC_0(R^+)$.

IV. Extensions of the Theory

Consider the equation

$$x'(t) = A(t)\{x(t) + (g_1x)(t)\} + \int_0^t B(t, s)\{x(s) + (g_2x)(s)\}\, ds + (g_3x)(t) + f(t), \qquad x(0) = x_0, \tag{11}$$

where g_i mapping $LL^1(R^+) \to LL^1(R^+)$ is a nonlinear functional. By the results in Section II, equation (11) is equivalent to the system

$$x(t) = R(t, 0)\, x_0 + \int_0^t R(t, s)\{f(s) + A(s)\, g_1x(s) + g_3x(s)\}\, ds + \int_0^t R(t, s)\left\{\int_0^s B(s, u)\,(g_2x)(u)\, du\right\} ds.$$

If (H1) is true, Fubini's theorem and equation (A) may be used to show that the last term of the equation above may be written in the form

$$\int_0^t \left\{\int_u^t R(t, s)\, B(s, u)\, ds\right\}(g_2x)(u)\, du = \int_0^t \left\{-\frac{\partial R(t, u)}{\partial s} - R(t, u)\, A(u)\right\} \times g_2x(u)\, du.$$

Thus (11) may be written in the equivalent form

$$x(t) = R(t, 0)\, x_0 + \int_0^t R(t, s)\{f(s) + A(s)(g_1x)(s) - A(s)(g_2x)(s) + (g_3x)(s)\}\, ds - \int_0^t \frac{\partial R(t, s)}{\partial s}(g_2x)(s)\, ds. \tag{12}$$

Let ρ be the map defined by Eq. (10) and let

$$\dot\rho(t) = -\int_0^t \frac{\partial R(t, s)}{\partial s}\varphi(s)\, ds \qquad (0 \leqslant t < \infty).$$

The method of proof used in Theorem 4 may then be applied to (12) in order to derive the following result:

Theorem 5. *Suppose that g_i is of higher order with respect to B for $i = 1, 2, 3$ and that* (H1) *is true and both ρ and $\dot\rho$ map $B \to B$. If $f \in B$ and $R(t, 0) \in B$, then for each $\epsilon > 0$ there exists an $\eta > 0$ such that if $|x_0| \leqslant \eta$ and $\|f\| \leqslant \eta$ then equation* (11) *has a unique solution x in B with $\|x\|_B \leqslant \epsilon$.*

V. An Application of Theorem

We shall now apply the results of Section III to study the stability problem for nuclear reactors. Following Gyftopoulos [10], we assume the equations governing the dynamic behavior of the reactor have the form

$$\gamma p'(t) = -\sum_{i=1}^{m} \beta_i\{p(t) - C_i(t)\} - v\int_0^t f(t-s)\,p(s)\,ds$$
$$- vp(t)\int_0^t f(t-s)\,p(s)\,ds, \tag{13}$$
$$C_i'(t) = \lambda_i\{p(t) - C_i(t)\} \qquad (i = 1, 2,\ldots, m)$$

where λ_i, β_i, γ and v are positive constants. The constant v is the designed steady state power of the reactor, $vp(t) + v$ the average reactor power at time t, and $C_i(t)$ the average density of the delayed neutrons of the i-th type. The function $f(t)$ is a reactivity feedback function which is assumed to be of class $L^1(R^+)$.

Equation (13) may be written in the vector form (N) if $x(t)$ is the column vector with components $p(t), C_1(t),\ldots, C_m(t)$, $B(t) = (b_{ij}(t))$ where $b_{11}(t) = vf(t)$ and $b_{ij}(t) \equiv 0$ otherwise, A is the matrix

$$\begin{bmatrix} -\sum_{i=1}^{m}\frac{\beta_i}{\gamma} & \frac{\beta_1}{\gamma} & \frac{\beta_2}{\gamma} & \frac{\beta_3}{\gamma} & \cdots & \frac{\beta_m}{\gamma} \\ \lambda_1 & -\lambda_1 & 0 & 0 & \cdots & 0 \\ \lambda_2 & 0 & -\lambda_2 & 0 & \cdots & 0 \\ \lambda_3 & 0 & 0 & -\lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \\ \lambda_m & 0 & 0 & & & -\lambda_m \end{bmatrix}$$

and $g\varphi$ is the column vector whose components are

$$(g_1\varphi)(t) = -v\varphi_1(t)\int_0^t f(t-s)\,\varphi(s)\,ds, \qquad \text{and} \qquad g_j\varphi(t) = 0$$

for $j = 2, 3,\ldots, m+1$.

The equations will be studied under the following assumption:

(H4) Let $f \in L^1(R^+)$ and let $f^*(s)$ denote the Laplace transform of f. Then for $\operatorname{Re} s \geqslant 0$ one has

$$D(s) = s + \left(\frac{s}{\gamma}\right)\sum_{i=1}^{m} \beta_i(s+\lambda_i)^{-1} + vf^*(s) \neq 0. \tag{14}$$

THEOREM 6, *Suppose* λ_i, β_i, γ, $v > 0$ *and* f *satisfy* (H4). *Then for each* $\epsilon > 0$ *there exists* $\eta > 0$ *such that if* $|P(0)|$, $|C_i(0)| \leqslant \eta$ *for* $i = 1, 2, \ldots, m$, *then the solution of equation* (13) *exists for all* $t \geqslant 0$, *is of class*

$$L^2(R^+) \cap BC_0(R^+), \qquad \textit{and} \qquad \sup_{0 \leqslant t < \infty} \left\{ |P(t)| + \sum_{i=1}^{m} |C_i(t)| \right\} \leqslant \epsilon.$$

Proof. Let

$$B = L^2(R^+) \cap BC_0(R^+)$$

with the norm $\| \varphi \|_B = \| \varphi \|_{L^2} + \| \varphi \|_{BC}$. Then B is a Banach subspace of $LL^1(R^+)$ with a stronger topology. It is easy to show that the functional g defined above is of higher order with respect to B. In order to apply Theorem 4 it remains to show that $R \in B$ and $\rho : B \to B$.

For this problem the matrix $\Psi(t, s) = \Psi(t - s)$ is of convolution type and is bounded; say $|\Psi(t)| \leqslant k$ for all $t \geqslant 0$. Thus equation (8) implies that

$$|R(t)| \leqslant 1 + \int_0^t |R(t-s)| \, |\Psi(s)| \, ds \leqslant 1 + k \int_0^t |R(s)| \, ds.$$

By Gronwall's inequality it follows that $|R(t)| \leqslant \exp(kt)$ for $t \geqslant 0$. Thus R has a Laplace transform $R^*(s)$ defined for $\operatorname{Re} s > k$. Since R satisfies equation (8), $R^*(s) = \{s - A - B^*(s)\}^{-1}$ if $\operatorname{Re} s > k$ and the inverse matrix exists. A straightforward calculation shows that $R^*(s) = H(s)/D(s)$ where $D(s)$ is defined by (14) and $H(s)$ is the matrix whose entries have the form

$$\begin{aligned}
H_{11}(s) &= 1, \\
H_{1,j+1}(s) &= \lambda_j (s + \lambda_j)^{-1} && (j = 1, 2, \ldots, m), \\
H_{j+1,1}(s) &= \frac{\beta_j}{\gamma} (s + \lambda_j)^{-1} && (j = 1, 2, \ldots, m), \\
H_{j+1,j+1}(s) &= \left\{ s + vf^*(s) + \sum_{i \neq j} \frac{\beta_i}{\gamma} \right\} (s + \lambda_j)^{-1} && (j = 1, 2, \ldots, m),
\end{aligned}$$

and for all other i and j

$$H_{i+1,j+1}(s) = (\beta_i \lambda_j) \, \gamma^{-1} (s + \lambda_i)^{-1} (s + \lambda_j)^{-1}.$$

The exact form of the matrix $H(s)$ is important only in the sense that it is clear that $H(s)$ exists as an analytic function for $\operatorname{Re} s > 0$, is continuous for $\operatorname{Re} s \geqslant 0$, and is bounded for $\operatorname{Re} s \geqslant 0$. Since (H4) is true it follows that $R^*(s)$ can be extended by analytic continuation to the half plane $\operatorname{Re} s > 0$ and is continuous in $\operatorname{Re} s \geqslant 0$.

As $|s| \to \infty$ with $\operatorname{Re} s \geqslant 0$, one has $vf^*(s) \to 0$ and

$$1 + \gamma^{-1} \sum_{i=1}^{m} \beta_i (s + \lambda_i)^{-1} \to 1.$$

Since $R^*(s)$ is continuous and bounded on compact subsets of the half plane $\operatorname{Re} s \geqslant 0$, it follows that there exists a constant $k > 0$ such that

$$|R^*(s)| \leqslant k(|s| + 1)^{-1} \qquad (\operatorname{Re} s \geqslant 0). \tag{15}$$

This implies the condition

$$\sup_{\sigma > 0} \left(\int_{-\infty}^{\infty} |R^*(\sigma + it)|^2 \, dt \right) < \infty.$$

By a theorem of Paley and Wiener [11, p. 8] it follows that $R \in L^2(R^+)$.

For any function $h \in L^2(R^+)$, the Laplace transform $(\rho h)^* (s) = R^*(s)\, h^*(s)$ exists and is analytic for $\operatorname{Re} s > 0$. Using (15) it follows that

$$\sup_{\sigma > 0} \int_{-\infty}^{\infty} |(\rho h)^* (\sigma + it)|^2 \, dt \leqslant k \sup_{\sigma > 0} \int_{-\infty}^{\infty} |h^*(\sigma + it)|^2 \, dt < \infty.$$

By the Paley-Weiner theorem $\rho h \in L^2(R^+)$. Since $(\rho h)(t) = \int_0^t R(t - s)\, h(s)\, ds$ is the convolution of two functions in $L^2(R^+)$ with $R \in C(R^+)$, ρh is also of class $BC_0(R^+)$. Thus ρ maps $B \to B$.

Since $R(t, s) = R(t - s)$ is of convolution type,

$$-\frac{\partial}{\partial s} R(t, s) = R'(t - s)$$

and

$$R'(t) = \left\{ A + \int_0^t B(\omega)\, d\omega \right\} + \int_0^t R'(t - s) \left\{ A + \int_0^s B(\omega)\, d\omega \right\} ds.$$

Therefore $R'(t)$ has Laplace transform equal to

$$\{s - A - B^*(s)\}^{-1} \{A + B^*(s)\}.$$

Since $|A + B^*(s)|$ is bounded on $\operatorname{Re} s \geqslant 0$, (15) and the Paley-Wiener theorems imply that $R' \in L^2(R^+)$. This means that $R(t)$ is uniformly Holder continuous with exponent $\frac{1}{2}$. Since $R \in L^2(R^+)$ and is uniformly continuous, $R(t) \to 0$ as $t \to \infty$ which implies that R is in B. An application of Theorem 4 completes the proof of Theorem 6. Q.E.D.

Theorem 6 gives local stability results for equation (13) under extremely mild assumptions on $f(t)$. In particular one need not assume that $f(t)$ is an exponential polynomial or other conditions which ensure that $f^*(s)$ has a meromorphic extension into the left-hand plane. Helliwell [12] has applied a different type of linearization to a slightly different nuclear reactor. He obtained similar but nonoverlapping results. Many authors have given sufficient conditions for the global asymptotic stability of Eqs. (13) or simpler versions of these equations; see Gyftopolous [10] for references. Their assumptions generally include a condition of the type (H4).

The ideas in the proof of Theorem 6 may be used to prove the following:

THEOREM 7. *Suppose* (H3) *is true.* $B \in L^1(R^+)$ *and* g_1 *and* g_2 *are of higher order with respect to the Banach space* $B = BC_0(R^+) \cap L^2(R^+)$. *Suppose* $\{s - A - B^*(s)\}^{-1}$ *exists and is bounded on* $\operatorname{Re} s \geqslant 0$ *with*

$$\sup_{\sigma \geqslant 0} \left\{ \int_{-\infty}^{\infty} |\{(\sigma + it) I - A - B^*(\sigma + it)\}^{-1}|^2 \, dt \right\} < \infty.$$

Then given any $\epsilon > 0$, *there exists* $\eta > 0$ *such that if* $|x_0| \leqslant \eta$, $f \in B$ *and* $\|f\| \leqslant \eta$, *the equation*

$$x'(t) = f(t) + Ax(t) + \int_0^t B(t-s)\{x(s) + (g_1 x)(s)\}\, ds + (g_2 x)(t), \qquad x(0) = x_0$$

has a unique solution $x \in BC_0(R^+) \cap L^2(R^+)$ *with* $\|x\| \leqslant \epsilon$.

Proof. The solution of this equation satisfies

$$x(t) = R(t)\, x_0 + \int_0^t R(t-s)\{f(s) + Ag_1 x(s) + g_3 x(s)\}\, ds + \int_0^t R'(t-s)\, g_1 x(s)\, ds.$$

As in the proof of Theorem 6, one shows that ρ maps $L^2(R^+) \to L^2(R^+)$, $R \in L_2(R^+)$ and $R' \in L^2(R^+)$. Since $(R')^*(s) = \{s - A - B^*(s)\}^{-1}(A + B^*(s)\}$ is bounded on $\operatorname{Re} s \geqslant 0$, the map $\dot{\rho}$ also takes $L^2 \to L^2$. Thus $R \in B$, $\rho : B \to B$ and $\dot{\rho} : B \to B$. An application of Theorem 5 completes the proof. Q.E.D.

VI. ON A PROBLEM OF LEVIN

The purpose of this section is to demonstrate some alternate techniques which are useful in the application of the theory in Sections II and III. We first show by example that the technique of proof used in Section V can be applied to certain integral equations whose kernels are not $L^1(R^+)$ (Theorems

8 and 9 below). Next, we show how the theory may be applied to study equations whose solutions are not necessarily $L^2(R^+)$ but have L^2 derivatives (Theorem 10).

J. J. Levin [13] studied certain problems of the form

$$x'(t) = -\int_0^t a(t-s)\,g(x(s))\,ds, \qquad x(0) = x_0\,, \tag{16}$$

where $g(x)$ is a nonlinear spring, $a(t) \in C[0, \infty) \cap C^3(0, \infty)$ and $(-1)^j a^{(j)}(t) \geqslant 0$ for $0 < t < \infty$ and $j = 0, 1, 2, 3$. He proved that all solutions of (16) exist for all $t \geqslant 0$ and tend to zero as $t \to 0$. Levin also studied the equation obtained by differentiating (16).

$$x''(t) = -a(0)\,g(x(t)) - \int_0^t a'(t-s)\,g(x(s))\,ds. \tag{17}$$

In this case he assumed that $g(x) = x$ is linear.

Consider the problem

$$x'(t) = k - \int_0^t a(t-s)\,x(s)\,ds, \qquad x(0) = 1. \tag{18}$$

Let $x_0(t)$ be the solution of (18) when $k = 0$ and $x_1(t)$ the solution when $k = 1$. Levin showed that the general solution of (17) has the form $x(t) = x_0(t)\,x(0) + x_1(t)\,x'(0)$. He also proved that if $a \in L^1(R^+)$, then $x_1(t) \to \{\int_0^\infty a(s)\,ds\}^{-1}$ as $t \to \infty$ and that if $a(\infty) > 0$, then $x_1(t) \to 0$ as $t \to \infty$. If $a(\infty) = 0$ but $a \notin L^1(R^+)$, he conjectured that $x(t) \to 0$.

Hannsgen [14] has recently generalized Levin's results for the linear case. He gave a rather complete analysis of the linear problem (18) when $k = 0$ or 1 under hypotheses on $a(t)$ which are weaker than Levin's. In particular he proves that Levin's conjecture is true. Following Hannsgen we assume that $a(t) = A(t) + C$ where

(L1) $A \in C(0, \infty) \cap LL^1(0, \infty)$; $A(t)$ is nonnegative and nonincreasing; $A(\infty) = 0$, $0 < A(0^+) \leqslant +\infty$; $C \geqslant 0$,

(L2) $A(t)$ is convex downward in $0 < t < \infty$, and

(L3) $A(t)$ admits no representation of the form

$$A(t) = \sum_{j=1}^{\infty} \delta_j \left\{1 - \frac{\min\{t, t_0\}}{jt_0}\right\},$$

where $t_0 = 2\pi/\tau_0$, $\delta_j \geqslant 0$, the set $\{j : \delta_j > 0\}$ has no common divisor bigger than 1, and $\{k + \sum_{j=1}^{\infty} \delta_j\}^{1/2} = j_0\tau_0$ for some integer j_0. Under these assumptions he proves:

LEMMA 4. *Suppose* $a(t) = A(t) + c$ *and* (LI-3) *are true. Let* $S = \{s : \operatorname{Re} s \geqslant 0, s \neq 0\}$. *Then*

(i) $A^*(s)$ *exists and is analytic for* $\operatorname{Re} s > 0$.

(ii) $A^*(s)$ *is continuous for* $s \in S$, $A^*(s) \to 0$ *as* $|s| \to \infty$, *and* $sA^*(s) \to 0$ *as* $|s| \to 0$, $s \in S$.

(iii) $c/s + A^*(s) + s \neq 0$ *if* $s \in S$.

(iv) $[s + A^*(s)]^{-1} \to 0$ *as* $|s| \to 0$ *if* $A \notin L^1(R^+)$.

(v) *The solution* $x(t)$ *of* (18) *exists for all* $t \geqslant 0$. *If* $c = 0$, $k = 1$ *and* $a \in L^1(R^+)$, *then* $x(t) \to \{\int_0^\infty a(s)\, ds\}^{-1}$ *as* $t \to \infty$. *If* $c > 0$ *or* $k = 0$ *or* $a \notin L^1(R^+)$, *then* $x(t) \to 0$.

Using Lemma 4 one can prove the following result.

LEMMA 5. *Suppose* (L1-3) *are true. Let* $B = BC_0(R^+) \cap L^2(R^+)$ *and let* $R(t)$ *be the resolvent of the linear equation* (18) *when* $k = 0$. *Then the operators* ρ *and* $\dot{\rho}$ *defined by the kernels* $R(t)$ *and* $R'(t)$ *map* $B \to B$ *and* $R \in B$.

Proof. Since $A^*(s) \to 0$ as $|s| \to \infty$ and

$$R^*(s) = \left\{s + A^*(s) + \frac{c}{s}\right\}^{-1}, (R')^*(s) = \left\{s + A^*(s) + \frac{c}{s}\right\}^{-1} \left\{A^*(s) + \frac{c}{s}\right\},$$

it follows that

$$|R^*(s)|, |(R')^*(s)| \leqslant k(|s| + 1)^{-1} \qquad (|s| \geqslant k_0) \tag{19}$$

for some constants k and k_0. If $c > 0$, then $R^*(s) = (s^2 + sA^*(s) + c)^{-1} s \to 0$ and $(R')^*(s) = (s^2 + sA^*(s) + c)^{-1}(sA^*(s) + c) \to 1$ as $|s| \to 0$. Then the estimate (19) (with a larger constant k) is true for all s with $\operatorname{Re} s \geqslant 0$. If $c = 0$ and $A \in L^1(R^+)$, then $R^*(s) \to \{\int_0^\infty A(t)\, dt\}^{-1}$ and $(R')^*(s) \to 1$ as $|s| \to 0$. If $c = 0$ and $A \notin L^1(R^+)$, then $R^*(s) \to 0$ as $|s| \to 0$. Moreover, estimates in Section 2 of Hannsgen's paper imply that

$$\begin{aligned} |(R')^*(\sigma + i\tau)| &\leqslant \left(\int_0^{\pi/2|\tau|} a(t) \exp(-\sigma t) \cos \tau t\, dt\right) \\ &\quad \times \left(-|s| + 2^{-1/2} \int_0^{\pi/2|\tau|} a(t) \exp(-\sigma t) \cos \tau t\, dt\right) \\ &\quad \to \sqrt{2} \qquad (|s| \to 0). \end{aligned}$$

Therefore the estimate (19) is also true for $\operatorname{Re} s \geqslant 0$ in the case $c = 0$.

Using (19) it follows as in the proof of Theorems 6 and 7 that ρ and

$\dot{\rho}$ map $L^2(R^+)$ into itself, $R \in L^2(R^+)$, $R' \in L^2(R^+)$, $R(t) \in BC_0(R^+)$, and ρ, $\dot{\rho}$ map $L^2(R^+) \to BC_0(R^+)$. Q.E.D.

Now consider the equation

$$x'(t) = -\int_0^t a(t-s)\{x(s) + g_1 x(s)\}\, ds + g_2 x(t), \qquad x(0) = x_0\,. \tag{20}$$

THEOREM 8. *Suppose* (L1-3) *are true and* g_1 *and* g_2 *are nonlinear functionals of higher order with respect to the space* $B = BC_0(R^+) \cap L^2(R^+)$. *Then given* $\epsilon > 0$, *there exists* $\eta > 0$ *such that if* $|x_0| \leqslant \eta$ *the solution* $x(t)$ *of* (20) *is in* B *and* $\|x\| \leqslant \epsilon$.

Proof. The result follows immediately from Lemma 5 and Theorem 5.

COROLLARY 1. *Suppose* (L1-3) *are true. If* $g(x) \in C^1$, $g(0) = 0$ *and* $g'(0) > 0$, *then given* $\epsilon > 0$, *there exists an* $\eta > 0$ *such that if* $|x_0| \leqslant \eta$ *the solution* $x(t)$ *of* (16) *exists for all* $t \geqslant 0$ *and* $|x(t)| \leqslant \epsilon$ *for all* t.

Proof. In this case $g(x) = g'(0)\, x + \{g(x) - g'(0)\, x\}$. Redefine $g(x)$ for x large in such a way so as to make g smooth and $g(x) - g'(0)\, x \equiv 0$. Then $g_1\varphi(t) = \{g\varphi(t) - g'(0)\, \varphi(t)\}$ is of higher order with respect to $BC_0(R^+) \cap L^2(R^+)$. The result then follows from Theorem 8. Q.E.D.

We shall now study the second order problem (17) where $g(x) = x + h(x)$ and $h(x)$ is of higher order. In addition to assumptions (L1-3), we shall need the following:

(L4) $a(t)$ is absolutely continuous on the interval $0 \leqslant t \leqslant T$ for each $T > 0$.

If one puts $x_1(t) = x(t)$ and $x_2(t) = x'(t)$, then equation (17) may be written in the vector form

$$x'(t) = A\{x(t) + Hx(t)\} + \int_0^t B(t-s)\{x(s) + Hx(s)\}\, ds, \qquad x(0) = x_0 \tag{21}$$

where

$$x(t) = \operatorname{col}(x_1(t), x_2(t)),$$

$$A = \begin{pmatrix} 0 & 1 \\ -a(0) & 0 \end{pmatrix}, \qquad B(t) = \begin{pmatrix} 0 & 0 \\ -a'(t) & 0 \end{pmatrix}, \qquad Hx = \begin{pmatrix} h(x_1) \\ 0 \end{pmatrix}.$$

Thus Eq. (21) has the form (11) with $g_1 x = g_2 x = Hx$, $g_3 \equiv 0$ and $f \equiv 0$. The resolvent for Eq. (21) has the form

$$R(t) = I + \int_0^t R(t-s)\, \Psi(s)\, ds, \qquad \Psi(t) = \begin{pmatrix} 0 & 1 \\ -a(t) & 0 \end{pmatrix}.$$

The variation of constants formula for (21) is

$$x(t) = R(t)\, x_0 + \int_0^t R'(t - s)\, Hx(s)\, ds. \tag{22}$$

Moreover, one has

$$R^*(s) = \{s + a^*(s)\}^{-1} \begin{pmatrix} 1 & s^{-1} \\ - a^*(s) & 1 \end{pmatrix} \tag{23}$$

and

$$(R'^*)(s) = \{s + a^*(s)\}^{-1} \begin{pmatrix} - a^*(s) & 1 \\ - sa^*(s) & - a^*(s) \end{pmatrix}. \tag{24}$$

THEOREM 9. *Suppose* (L1-4) *hold and* $c > 0$. *If* $g(x) = x + h(x)$ *where* $h \in C^1$ *and* $h(0) = h'(0) = 0$, *then given* $\epsilon > 0$, *there exists* $\eta > 0$ *such that if* $|x(0)| + |x'(0)| \leqslant \eta$, *then the solution* $X(t) = (x(t), x'(t))$ *of* (17) *exists for all* $t \geqslant 0$, $X \in BC_0(R^+) \cap L^2(R^+)$ *and* $\| X(\cdot)\| \leqslant \epsilon$.

Proof. Denote the entries of $R(t)$ by R_{ij} for $i, j = 1, 2$. As in the proof of Lemma 5 it follows from (23) and (24) that R_{11}, R_{12}, R_{21}, and R_{22} are in $L^2(R^+)$ and the operators defined by these kernels map the space $B = BC_0(R^+) \cap L^2(R^+)$ into itself. Moreover $R' \in L^2(R^+)$ and maps $B \to B$. Lemma 4 implies that $R_{ij} \in BC_0(R^+)$ for $i, j = 1, 2$.

Since $H\varphi$ is of higher order with respect to B, Theorem 5 may now be applied. Q.E.D.

If $c = 0$ then different arguments must be used. Let B_l be the set of column vectors $\varphi = (\varphi_1, \varphi_2)$ such that $\lim \varphi_1(t)$ exists as $t \to \infty$, $\lim \varphi_2(t) = 0$ as $t \to \infty$ and $\varphi_1', \varphi_2' \in L^2(R^+)$. For any such $\varphi(t)$ define a norm

$$\| \varphi \| = \sup\{| \varphi(t)| : t \geqslant 0\} + \left\{\int_0^\infty | \varphi'(t)|^2\, dt\right\}^{1/2}.$$

Then B_l is a Banach space whose topology is stronger than $LL^1(R^+)$. Let B_0 be the subspace of B_l such that $(\varphi_1, \varphi_2) \in B_0$, then $\varphi_1(\infty) = 0$.

THEOREM 10. *Let* (L1-4) *hold with* $c = 0$ *and* $a \in L^1(R^+)$. *Suppose* B_l *is the space defined above. If* $g(x) = x + h(x)$ *where* $h \in C^1$ *and* $h(0) = h'(0) = 0$, *then given* $\epsilon > 0$ *there exists* $\eta > 0$ *such that if* $| X_0 | \leqslant \eta$, *then the solution* $X(t) = (x_1(t), x_2(t))$ *of* (22) *exists for all* $t \geqslant 0$, *belongs to* B_l, *and* $\| X(\cdot)\| \leqslant \epsilon$. *Moreover,*

$$x_1(\infty) = x_2(0)\Big/\int_0^\infty a(s)\, ds - h(x_1(\infty)).$$

If $c \neq 0$ *or* $a \notin L^1(R^+)$, *then the same result holds with* $x_1(\infty) = 0$.

Proof. From Lemma 4 and the proof of Theorem 9 it is known that $R' \in L^2$, $|R_{11}(t)| + |R_{21}(t)| + |R_{22}(t)| \to 0$ as $t \to \infty$, $R_{12}(t) \to \{\int_0^\infty a(s)\, ds\}^{-1}$ as $t \to \infty$ and R_{11}, $R_{21} \in L^2(R^+)$. Consider equation (22) under the hypotheses of this theorem. The remarks above verify that $R(t) X_0 \in B_l$. It is easy to show that $H\varphi$ is of higher order with respect to B_l.

The map $\int_0^t R'(t-s) H\varphi(s)\, ds$ has the form

$$\operatorname{col}\left(\int_0^t (R_{11}')(t-s)\, h(\varphi(s))\, ds, \int_0^t R_{21}'(t-s)\, h(\varphi(s))\, ds\right).$$

Therefore it remains to show that if $\Phi = \operatorname{col}(\varphi_1, \varphi_2)$, then the map

$$\rho_0 \Phi(t) = \operatorname{col}\left(\int_0^t R_{11}'(t-s)\, \varphi_1(s)\, ds, \int_0^t (R_{21}')(t-s)\, \varphi_1(s)\, ds\right) \text{ maps } B_l \to B_l$$

Let $\Phi(t) \in B_l$. Since $R_{11} \in BC_0(R^+) \cap L^2(R^+)$ and $(\varphi_1') \in L^2$,

$$\begin{aligned}\int_0^t (R_{11}')(t-s)\, \varphi_1(s)\, ds &= -R_{11}(t-s)\, \varphi_1(s)\,\Big|_0^t + \int_0^t R_{11}(t-s)\, (\varphi_1')(s)\, ds \\ &= R_{11}(t)\, \varphi_1(0) - \varphi_1(t) + \int_0^t R_{11}(t-s)\, \varphi_1'(s)\, ds \\ &\to 0 \cdot \varphi_1(0) - \varphi_1(\infty) + 0\end{aligned}$$

as $t \to \infty$. Moreover, since $R'_{11} \in L^2(R^+)$ and maps $L^2 \to L^2$, one has

$$\begin{aligned}\frac{d}{dt}\left(\int_0^t R_{11}'(t-s)\, \varphi_1(s)\, ds\right) &= \frac{d}{dt}\left(R_{11}(t)\, \varphi_1(0) - \varphi_1(t) + \int_0^t R_{11}(t-s)\, \varphi_1'(s) ds\right) \\ &= R_{11}'(t)\, \varphi_1(0) - \varphi_1'(t) + R_{11}(0)\, \varphi_1'(t) \\ &\quad + \int_0^t R_{11}'(t-s)\, \varphi_1'(s)\, ds \\ &= R_{11}'(t)\, \varphi_1(0) + \int_0^t R_{11}'(t-s)\, \varphi_1'(s)\, ds \in L^2.\end{aligned}$$

Similarly, the function

$$\int_0^t R_{21}'(t-s)\, \varphi_1(s)\, ds = R_{21}(t)\, \varphi_1(0) + \int_0^t R_{21}(t-s)\, \varphi_1'(s)\, ds$$

tends to zero as $t \to \infty$ and has an L^2 derivative. Therefore ρ_0 maps $B_l \to B_l$ and hence, if $|X_0|$ is small, the right hand side of equation (22) defines a contraction mapping on small neighborhoods of the origin in B_l. The last assertion is proved in the same manner with B_l replaced by B_0. Q.E.D.

Notice that in Theorem 9 the hypothesis $h \in C^1(R^+)$, $h(0) = h'(0) = 0$ could be weakened considerably. Indeed in Eq. (21) replace the term $AHx(t)$ by a functional $g_1X(t)$; replace the term $Hx(s)$ under the integral by a functional $g_2X(s)$ and add to the right side a vector valued function $f(t)$. If $f \in B$, $\| f \|$ is small, and g_1, g_2 are of higher order with respect to B, then Theorem 9 remains true. In Theorem 10 the functional $h(\varphi(t))$ must have an L^2 derivative if $\| \varphi \|$ is small. Thus h must be C^1 near $x = 0$.

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